Discrete geodesics

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Abstract. The article deals with the discrete surfaces and discrete differential geometry. Three types of discrete geodesics are introduced, their characteristics and differences are described. The application of discrete geodesics for the shortest path problem will be discussed.

Keywords: discrete differential geometry, geodesic, polyhedral model, data-dependant triangulation, the shortest path problem, Gauss curvature

1 Introduction

The motivation to this research spring up from the former not-yet fully solved The Shortest Path problem.

The input to the shortest path problem is a polyhedral surface and two points s and t on it. The task is to find the shortest path between s and t that goes along the surface. A geodesic can be used for the shortest path problem while a theorem from differential geometry [3] says: if there is the shortest path between two points on the surface then it is the geodesic.

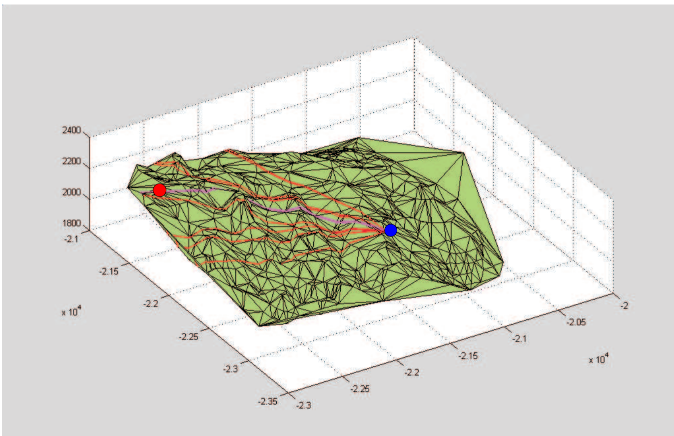

Figure 1: The shortest path problem solved by geodesics

In author's former article [5] an algorithm is introduced, that partly solves the shortest path problem by geodesic. The difficulties consist in the fact that the algorithm works well only for some types of surfaces.

This article will answer the question why it is not easy to compute the geodesic on some types of discrete surfaces.

To answer this question it is necessary to deal with the geometry of discrete surfaces and with the types of discrete surfaces. The following section will describe some basic facts of the discrete differential geometry and will concentrate especially on the discrete geodesics.

2 Discrete differential geometry

The discrete differential geometry describes the (differential) geometry of discrete surfaces. Besides others it defines the curvatures and geodesics on the discrete surfaces.

Author supposes that the reader has basic knowledge of differential geometry ([3]).

2.1 Discrete surfaces

The discrete surface is a polyhedral model of a smooth surface, reconstructed from the given discrete data set, decomposed in triangles, quadrilaterals or other polygons. Model is *adequate* if the geometric characteristic of the source smooth model can be determined by computing its curvatures.

There are two examples of triangulation on the figure 2. The left one is the Delaunay triangulation, the right one is the tightest triangulation. Which of them does characterize the smooth source surface better?

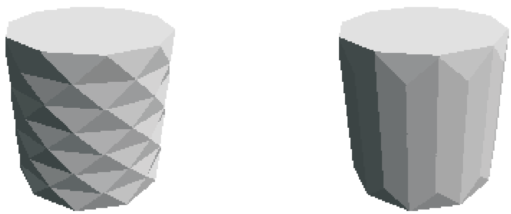

Figure 2: Two examples of triangulation. Left: Delaunay triangulation; Right: Tightest triangulation. Adapted from [1]

2.2 Discrete curvatures

The definition of mean, Gauss and geodetic curvature can be found e.g. in [1], [4], [2]. The following paragraph is the summary of that resources. Because of the limited text we will concentrate here only on the Gauss and geodesic curvature and discrete geodesics.

Definition 1 *The Gauss curvature on the smooth surfaces is the number* $\kappa = \kappa_1 \cdot \kappa_2$*, where* κ_1 *and* κ_2 *are the values of the minimal and maximal normal curvatures in the given point.*

With the knowledge of the value of Gauss curvature (especially its sign) can be estimated the type of the surface in given point (ridge, valley, saddle etc.).

On the discrete surface the Gauss curvature is concentrated only in the vertices.

Definition 2 *Let* V *be a vertex of a polyhedral surface* S *with the total vertex angle* α_t*. The total Gauss curvature* $\kappa(V)$ *of a vertex* V *is defined as*

$$\kappa(V) = 2\pi - \alpha_t \tag{1}$$

Depending on α_t the vertices can be classified into three groups:

1. *The spherical vertices:* $\alpha_t < 2\pi$. After unfolding its neighbourhood does not fill in the whole neighbourhood of the unfolded vertex in the Euclidean plane.
2. *The Euclidean vertices:* $\alpha_t = 2\pi$. After unfolding its neighbourhood fill in the whole neighbourhood of the unfolded vertex in the Euclidean plane.
3. *The hyperbolic vertices:* $\alpha_t > 2\pi$. After unfolding its neighbourhood overlaps themselves.

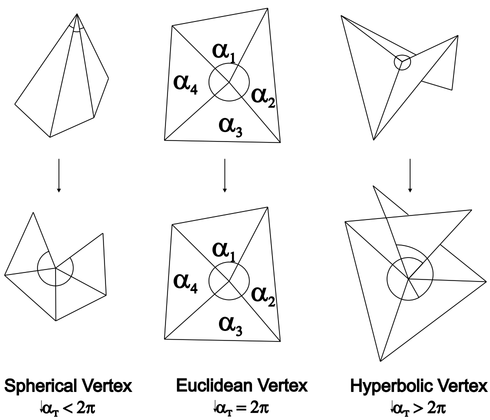

Figure 3: Three types of vertices

The geodesic curvature is a known term from the differential geometry.

Definition 3 *Let* $\mathbf{P}(\mathrm{t}), \mathrm{t} \in \jmath$ *be the curve on the surface. The size of the perpendicular projection of the vector of the first curvature* $\ddot{\mathbf{P}}$ *of the curve on the surface on to the tangent plane is called the geodesic curvature.* $\jmath$ *is an open interval.*

Theorem 1 *For the geodesic curvature holds* $k^g = |(\mathbf{n}, \dot{\mathbf{P}}, \ddot{\mathbf{P}})|$ *where* $\mathbf{n}$ *is the unit normal vector of the surface,* $\dot{\mathbf{P}}$ *is the unit tangent vector of the curve and* $\ddot{\mathbf{P}}$ *is the vector of the first curvature of the curve.*

Proof can be constructed from the geometrical sence of the mixed product.

Similarly as the geodesic curvature is defined on smooth surfaces, the *discrete geodesic curvature* is defined on polyhedral surfaces. The discrete geodesic curvature of a curve is the normalized angle between the curve and the discrete straightest geodesic γ.

Definition 4 *Let* γ *be a curve on a polyhedral surface* S*. Let* α_t *be the total vertex angle and* α_r *one of the two curve angles of* γ *at a point* P*. The discrete geodesic curvature* κ_g *of* γ *at* P *is given by*

$$\kappa_g = \frac{2\pi}{\alpha_t}\left(\frac{\alpha_t}{2} - \alpha_r\right). \tag{2}$$

Choosing the other curve angle changes the sign of κ_g*.*

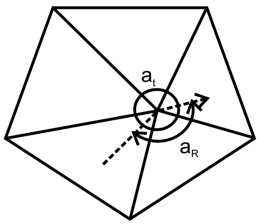

Figure 4: Geodesic curvature

2.3 Discrete geodesics

The geodesic on the smooth surface is uniquely determined.

Definition 5 *J is a geodesic if one of the following properties holds:*

1. *J is locally the shortest curve*
2. *J has vanishing geodesic curvature* $\kappa_g = 0$
3. *J is parallel to the surface normal*

Unfortunately there is no similar theorem which would hold for the discrete surfaces. That's why there is more types of discrete geodesics: If the geodesic fulfils the first condition then it is called *the shortest geodesic*. If the geodesic fulfils the second condition then it is called *the straightest geodesic*. There is no equivalent name for geodesics that fulfil the third condition. Throughout such geodesics can be found and they differ from the others in the hyperbolic vertices.

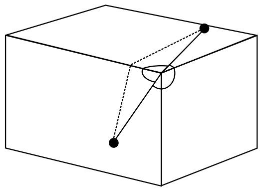

Figure 5: Two types of discrete geodesics. Both of them are the straightest geodesics but only the left one is the shortest geodesic.

2.3.1 How do the discrete geodesics differ?

1. In the triangles, edges and Euclidean vertices they are the same.
2. In the spherical vertex: The shortest geodesic never goes through the spherical vertex (see figure 5).
3. In the hyperbolic vertex two following theorems hold.

Theorem 2 *There is a set of the shortest geodesics coming in a hyperbolic vertex with the same inbound direction. Only one of them goes on with the same direction as a straightest geodesic.*

Proof 1 *It suffices to realize that the geodesics which have the angle bigger than π on both sides cannot be locally shortened.*

Theorem 3 *The straightest geodesics do not solve the boundary value problem for geodesics since there exist shadow regions in the neighbourhood of a hyperbolic vertex where two points cannot be joined by the straightest geodesic.*

Proof 2 *The theorem can be easily proofed after cutting the hyperbolic vertex along the input part of the geodesic and development into the plane.*

3 Conclusion

Now we can answer the key question: How do the discrete surfaces affect the solution of the shortest path problem?

1. The shortest path cannot go through the spherical vertices (except of its first and last points).
2. The straightest geodesic is impractical for the shortest path problem because of its characteristics in the neighbourhood of the hyperbolic vertices.
3. The success depends also on the precise choice of the appropriate triangulation (on the type of the discrete surface).

The future work will concentrate on the following problems:

- What is the best (data-dependent) triangulation for the shortest path problem?
- Can the third type of geodesic be useful for the shortest path problem?
- Can we estimate the area where the shortest path will lead if we know the curvature of the surface?

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