Generalization of Laguerre Geometry

Jaromír Dobrý

Department of Mathematics, University of West Bohemia
Univerzitní 22, 306 14 Plzeň
dobry@kma.zcu.cz

Abstract. The article deals with the relationship between Laguerre geometry and Minkowski sum. This relationship is used to create new geometry based generally on fundamental objects different to closed balls that formed Laguerre geometry. In the end of the article, there is defined also generalization of medial axis transform to the newly defined spaces and proved its uniqueness. The article continues the same called article from Proceedings of 25th Conference on Geometry and Computer Graphics, 2005

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1 Introduction

We will study the relationship between the operation of Minkowski Sum and Laguerre geometry. To make this possible, we will consider rather closed balls as solids than spheres as surfaces.

Definition 1 *Consider a linear space* $\mathbf{V}$ *over the field* $\mathbb{R}$, $\mathbf{A}, \mathbf{B} \subseteq \mathbf{V}$, $\lambda \in \mathbb{R}$, $\lambda \geq 0$. *The set* $\mathbf{A} + \mathbf{B} = \{x + y : x \in \mathbf{A}, y \in \mathbf{B}\}$ *we call* Minkowski sum *of* $\mathbf{A}$ *and* $\mathbf{B}$, $\lambda \cdot \mathbf{A} = \{\lambda x : x \in \mathbf{A}\}$ *we call* λ-multiple *of* $\mathbf{A}$.

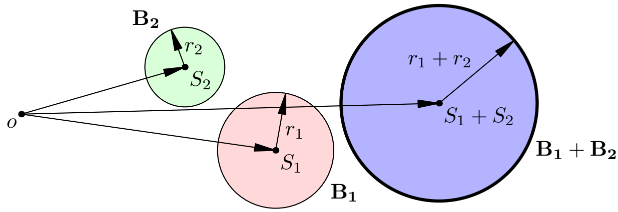

Figure 1: Minkowski sum of closed balls

In [1] there was introduced that two relatively different structures behave in a very simmilar way. The first structure was $(\mathfrak{B}, +, \cdot)$ (see figure 1), set of all closed balls in inner product space with operations of Minkowski Sum and of non-negative multiple, the second $(\mathbb{R}_0^+, +, \cdot)$ (see figure 2), non-negative real numbers with commonly defined operations. There was also introduced a way to add inverse elements to $(\mathfrak{B}, +, \cdot)$ so that it forms a linear space.

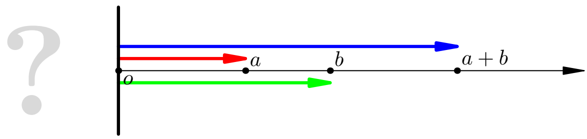

Figure 2: Structure $(\mathbb{R}_0^+, +, \cdot)$

2 Building the space

The first difference from last year is the next definition. The set of sufficient conditions to build a partially ordered linear space. It turned out, that these conditions were too strong for our purposes, they have been a little weakened.

Definition 2 *Let $\mathcal{P}(\mathbf{V})$ denote the power set of the linear space $\mathbf{V}$. Denote by $\mathfrak{M}$ any subset of $\mathcal{P}(\mathbf{V})$ for which following conditions are satisfied:*

(M1) $\forall x \in \mathbf{V} : \{x\} \in \mathfrak{M}$

(M2) $\forall \mathbf{A}, \mathbf{B} \in \mathfrak{M} : \mathbf{A} + \mathbf{B} \in \mathfrak{M}$

(M3) $\forall \lambda \in \mathbb{R}_0^+, \mathbf{A} \in \mathfrak{M} : \lambda \mathbf{A} \in \mathfrak{M}$

(M4) $\forall \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathfrak{M} : \mathbf{A} + \mathbf{C} \subseteq \mathbf{B} + \mathbf{C} \Rightarrow \mathbf{A} \subseteq \mathbf{B}$

(M5) $\forall \lambda, \alpha \in \mathbb{R}_0^+, \mathbf{A} \in \mathfrak{M} : (\lambda + \alpha)\mathbf{A} = \lambda \mathbf{A} + \alpha \mathbf{A}$

Remark 1 Note that **(M4)** implies the cancelation property. In general, $(\mathfrak{M}, +)$ is a monoid with cancelation property, For non-negative real numbers both $+, \cdot$ distribution laws hold, for all $\mathbf{A} \in \mathfrak{M}$: $1 \cdot \mathbf{A} = \mathbf{A}$ and inverse elements don't generally exist. The non-existence of inverse elements is the only reason why $(\mathfrak{M}, +, \cdot)$ is not a linear space, but they can be easily added to this space.

It may be proved, that these conditions are sufficient to form a partially ordered linear space $\mathcal{V}(\mathfrak{M})$ in exactly the same way as in [1]. The set of oriented balls defined in [1] is a particular example of this space and is "identical" to the cyclographic model of Laguerre geometry. We will also use the canonical injection $\varphi : \mathfrak{M} \to \mathcal{V}(\mathfrak{M})$.

3 Alexandrov topology

In [1], there was defined the generalized subset relation to describe properties like "be contained in." This relation can be also described in terms of topology.

Definition 3 *Let* $\mathbf{X} = (\mathbf{X}, \preceq)$ *be a preordered set. Define open sets:*

$$\tau = \{\mathbf{S} \subseteq \mathbf{X} : \forall x, y \in \mathbf{X} : x \in \mathbf{S} \wedge x \preceq y \Rightarrow y \in \mathbf{S}\}$$

If we take τ *as set of all open sets, we obtain a topology, that is called* Alexandrov topology*.*

Remark 2 Closed sets in Alexandrov topology are down-sets:

$$\{\mathbf{S} \subseteq \mathbf{X} : \forall x, y \in \mathbf{X} : y \in \mathbf{S} \wedge x \preceq y \Rightarrow x \in \mathbf{S}\}$$

We may define Alexandrov topology on space $\mathcal{V}(\mathfrak{M})$ equipped with generalized subset relation. Closed set in this topology is, in other words, the set $\mathbf{A}$, which contains with every element $X \in \mathbf{A}$ also all elements Y contained in X. The mappings that preserve generalized subset relation may be interpretted as a kind of generalization of Laguerre mappings and following theorem allows us to describe them also as homeomorphisms in Alexandrov topology.

Theorem 1 *Let* $(\mathbf{X}, \preceq)$ *be a partially ordered set,* τ *its Alexandrov topology. Isomorphisms of order relation* $\preceq$ *are exactly the homeomorphisms in topology* τ*.*

4 Example – One-set Generated Space

In this section, we will show a particular kind of the new geometry generated by one set with all its scales and translations.

Definition 4 *Let* $\mathbf{V}$ *be an inner product space,* $\mathbf{C} \subseteq \mathbf{V}$ *be bounded convex set,* $o \in \mathbf{C}$*. Define a mapping* $r_{\mathbf{C}} : \mathbf{V} \to (\mathbb{R}_0^+ \cup \{+\infty\})$*,* $r_{\mathbf{C}}(o) = \infty$*,* $r_{\mathbf{C}}(u) = \sup\{r \in \mathbb{R}_0^+ : \frac{ru}{\|u\|} \in \mathbf{C}\}$ *for* $u \neq o$*. Image* $r_{\mathbf{C}}(u)$ *will be called* radius in direction u *and mapping* $r_{\mathbf{C}}$ *itself will be called* radius in direction*.*

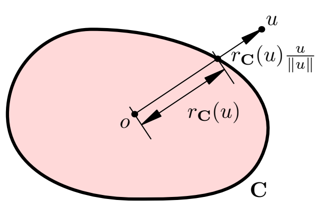

Figure 3: Radius in direction u

Lemma 1 *Let* $\mathbf{V}$ *be an inner product space,* $\mathbf{C} \subseteq \mathbf{V}$ *be a bounded convex set,* $o \in \mathbf{C}$*,* $\mathbf{C} \neq \{o\}$*. Then for every* $\mathbf{C}_1 = \lambda_1\mathbf{C} + \{x_1\}$*,* $\mathbf{C}_2 = \lambda_2\mathbf{C} + \{x_2\}$ *we have*

- $\mathbf{C}_1 \subseteq \mathbf{C}_2 \Rightarrow \lambda_1 \leq \lambda_2$
- $\mathbf{C}_1 \subseteq \mathbf{C}_2 \Leftrightarrow (\lambda_1 \leq \lambda_2) \wedge \Big(\|x_1 - x_2\| \leq r_{\mathbf{C}}(x_1 - x_2) \cdot (\lambda_2 - \lambda_1)\Big)$
- $\mathbf{C}_1 = \mathbf{C}_2 \Leftrightarrow (\lambda_1 = \lambda_2) \wedge (x_1 = x_2)$

For our purposes, define $\infty \cdot x = \infty$ *for all* $x \in \mathbb{R} \cup \{\infty\}$*.*

The lemma gives us the analytical description of the subset relation. We may consider all translations and scales of the fundamental object to be the set $\mathfrak{M}$ from definition 2.

Lemma 2 *Let* $\mathbf{V}$ *be an inner product space,* $\mathbf{C} \subseteq \mathbf{V}$ *be a bounded convex set,* $o \in \mathbf{C}$*,* $\mathbf{C} \neq \{o\}$*. Define a set*

$$\mathfrak{M}(\mathbf{C}) = \Big\{\lambda\mathbf{C} + \{x\} : \lambda \in \mathbb{R}_0^+, x \in \mathbf{V}\Big\}$$

Then $(\mathfrak{M}(\mathbf{C}), +, \cdot)$ *is isomorphic to* $(\mathbb{R}_0^+ \times \mathbf{V}, +, \cdot)$ *with coordinate-wise defined operations.*

Proof: The proof is straightforward.

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Theorem 2 *Let* $\mathbf{V}$ *be an inner product space,* $\mathbf{C} \subseteq \mathbf{V}$ *be a bounded convex set,* $o \in \mathbf{C}$*,* $\mathbf{C} \neq \{o\}$*. Then* $\mathfrak{M}(\mathbf{C})$ *satisfies conditions of definition 2.*

Proof: We have:

- **(M1)**: Obviously $0 \cdot \mathbf{C} + \{x\}$ is required element.
- **(M2)**,**(M3)**: Follows immediately from lemma 2
- **(M4)**: By lemma 2 and cancelation property of $\mathbb{R}_0^+$ and $\mathbf{V}$.
- **(M5)**: By lemma 2 and distributivity of $\mathbb{R}_0^+$ and $\mathbf{V}$.

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Definition 5 *Let* $\mathbf{V}$ *be an inner product space,* $\mathbf{C} \subseteq \mathbf{V}$ *be a bounded convex set,* $o \in \mathbf{C}$*,* $\mathbf{C} \neq \{o\}$*, then the space* $\mathbf{V}(\mathfrak{M}(\mathbf{C}))$ *will be called* oriented set space generated by $\mathbf{C}$.

Remark 3 Oriented ball space is particular example of one-set generated oriented set space, setting $\mathbf{C} = B(o, 1)$, the closed unit ball.

Following theorem gives us an equivalent description to the cyclographic model of Laguerre geometry. The only difference is the use of generalized subset relation instead of indefinite inner product.

Theorem 3 *Let* $\mathbf{V}(\mathfrak{M}(\mathbf{C}))$ *be oriented set space generated by* $\mathbf{C}$. *Define mapping* $\alpha : \mathbf{V}(\mathfrak{M}(\mathbf{C})) \to \mathbb{R} \times \mathbf{V}$,

$$\alpha([\lambda_1 \mathbf{C} + \{x_1\}, \lambda_2 \mathbf{C} + \{x_2\}]) = [\lambda_1 - \lambda_2, x_1 - x_2].$$

Then α *is an isomorphism of* $(\mathbf{V}(\mathfrak{M}(\mathbf{C})), +, \cdot)$ *onto* $(\mathbb{R} \times \mathbf{V}, +, \cdot)$ *with coordinate-wise defined operations. Generalized subset relation* $\preceq$ *may be written in terms of space* $\mathbb{R} \times \mathbf{V}$*:*

$$[\lambda_A, x_A] \preceq [\lambda_B, x_B] \Leftrightarrow (\lambda_A \leq \lambda_B) \wedge \Big(\|x_A - x_B\| \leq r_{\mathbf{C}}(x_A - x_B) \cdot (\lambda_B - \lambda_A) \Big)$$

5 Set representation

In this section we define mappings that are useful for representation of arbitrary set $\mathbf{X} \subseteq \mathbf{V}$. The most powerful tool will be a mapping GMAT which may be interpreted as generalized MAT – medial axis transform.

Definition 6 *Define mapping* $\mathcal{R} : \mathcal{P}(\mathbf{V}) \to \mathcal{P}(\mathcal{V}(\mathfrak{M}))$*:*

$$\mathcal{R}(\mathbf{A}) = \{X \in \mathcal{V}(\mathfrak{M}) : \exists \mathbf{Y} \in \mathfrak{M} : X \preceq \varphi(\mathbf{Y}) \wedge \mathbf{Y} \subseteq \mathbf{A}\}$$

Remark 4 For any set $\mathbf{A}$ a set $\mathcal{R}(\mathbf{A})$ will be a down set in generalized subset relation and therefore closed considering Alexandrov topology.

Definition 7 *Let* $\mathbf{A} \subseteq \mathcal{V}(\mathfrak{M})$ *be down set. Let* $\mathrm{GMAT}(\mathbf{A}) \subseteq \mathbf{A}$ *be a set satisfying following conditions:*

(GMAT1) $\forall X, Y \in \mathrm{GMAT}(\mathbf{A}), X \neq Y : X \npreceq Y \wedge Y \npreceq X$

(GMAT2) $\forall X \in \mathbf{A} : \exists Y \in \mathrm{GMAT}(\mathbf{A}) : X \preceq Y$

The set $\mathrm{GMAT}(\mathbf{A})$ *will be called* generalized medial axis transform.

Lemma 3 (equivalent definition) *Let* $\mathbf{A} \subseteq \mathcal{V}(\mathfrak{M})$ *be closed in Alexandrov topology. The set* $\mathrm{GMAT}(\mathbf{A}) \subseteq \mathbf{A}$ *is generalized medial axis transform if and only if it satisfies following conditions:*

(GMAT1') $\mathrm{GMAT}(\mathbf{A})$ *as topological subspace of* $\mathcal{V}(\mathfrak{M})$ *is discrete*

(GMAT2') $\mathrm{cl}\mathrm{GMAT}(\mathbf{A}) = \mathbf{A}$

Proof: Obviously **(GMAT1)** and **(GMAT1')** are equivalent, so are **(GMAT2)** and **(GMAT2')**.

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Remark 5 If $\mathrm{GMAT}(\mathbf{A})$ exists, it is exactly the set of all maximal elements of $\mathbf{A}$. From **(GMAT2)**, every maximal element must be in $\mathrm{GMAT}(\mathbf{A})$, from **(GMAT1)** any element contained in maximal element must not be in $\mathrm{GMAT}(\mathbf{A})$. If there is X for which there is no maximal element $Y \in \mathbf{A} : X \preceq Y$ then $\mathrm{GMAT}(\mathbf{A})$ doesn't exist.

Remark 6 Consider an oriented ball space $\mathcal{V}(\mathfrak{B})$ upon the inner product space $\mathbf{V}$. Let $\mathbf{A} \subseteq \mathbf{V}$ be a closed bounded set. Then $\mathrm{GMAT}(\mathcal{R}(\mathbf{A}))$ becomes an ordinary medial axis transform.

Lemma 4 *Let* $\mathbf{A} \subseteq \mathcal{V}(\mathfrak{M})$*, let* $\mathrm{GMAT}(\mathbf{A})$ *exist. Then* $\mathrm{GMAT}(\mathbf{A})$ *is uniquely determined.*

Proof: Consider we have two different $\mathrm{GMAT}_1(\mathbf{A})$, $\mathrm{GMAT}_2(\mathbf{A})$. Without loss of generality assume there is a $X \in \mathrm{GMAT}_1(\mathbf{A})$, $X \notin \mathrm{GMAT}_2(\mathbf{A})$. Obviously $X \in \mathbf{A}$, from **(GMAT2)** there is $Y \in \mathrm{GMAT}_2(\mathbf{A})$, $X \preceq Y$, $X \neq Y$, and obviously $Y \in \mathbf{A}$. But from **(GMAT2)** we have some $Z \in \mathrm{GMAT}_1(\mathbf{A})$, $X \preceq Y \preceq Z$, $Z \neq X$ a contradiction with **(GMAT1)**. ■

6 Conclusion and Future Work

We have formulated sufficient conditions to build space similar to oriented sphere space from Laguerre geometry, but more general. Space of oriented spheres (exactly oriented balls) is particular example of these more general spaces. List of conditions has been reduced to 5 relatively easily verifiable conditions. The spaces are translation-invariant (if the set is an element of the space, then also arbitrary translated set is an element of it). The definition of MAT was also generalized to these kind of spaces.

In my future work, I will concentrate on spaces that are also rotation-invariant, the subclass of the spaces built in these two articles. They may be useful to describe 5-axis milling and perhaps to solve today's problems of CAGD.

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