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GENERALIZATION OF LAGUERRE GEOMETRY

Abstract

The article deals with the problem of building fundamentals of Laguerre geometry using Minkowski sum. The space of Laguerre's oriented spheres is shown as a result of an effort to get space that forms a group with Minkowski sum. If we start with a set of closed balls in inner product space, we get a space with behaviour identical to classical oriented sphere space. This way we show that Laguerre sphere geometry is a particular example of more general kind of geometry based on operation of Minkowski sum.

Keywords

Laguerre geometry, oriented sphere, oriented contact, closed ball, Minkowski sum, Klein geometry, transformation, group, linear space

1 Building the space

Definition 1 *Consider a linear space* $\mathbf{V}$ *over the field* $\mathbb{R}$, $\mathbf{A}, \mathbf{B} \subset \mathbf{V}$, $\lambda \in \mathbb{R}$, $\lambda \geq 0$. *The set* $\mathbf{A} + \mathbf{B} = \{x + y : x \in \mathbf{A}, y \in \mathbf{B}\}$ *we call* Minkowski sum *of* $\mathbf{A}$ *and* $\mathbf{B}$, $\lambda \cdot \mathbf{A} = \{\lambda x : x \in \mathbf{A}\}$ *we call* λ-multiple *of* $\mathbf{A}$.

Example 1 Let $\mathbf{X}$ be an inner product space over the field $\mathbb{R}$, denote by $\mathfrak{B}$ the set of all closed balls in $\mathbf{X}$. Then $(\mathfrak{B}, +)$ is a monoid with the cancellation property, i.e. $+$ is closed on $\mathfrak{B}$, commutative, asociative, $(\mathfrak{B}, +)$ has the identity element $\{o\}$ and the cancellation property. For non-negative real numbers both $+, \cdot$ distribution laws hold, $1 \cdot \mathbf{A} = \mathbf{A}$ for any $\mathbf{A} \in \mathfrak{B}$, but inverse elements to $+$ don't generally exist. The non-existence of inverse elements is the only reason why $(\mathfrak{B}, +, \cdot)$ is not a linear space.

□

Example 2 Let's consider the set of all non-negative real numbers $\mathbb{R}_0^+$ and operations $+, \cdot$ upon it. Then $(\mathbb{R}_0^+, +)$ is a monoid with the cancellation property. Both $+, \cdot$ distribution laws hold, $1 \cdot x = x$ for any $x \in \mathbb{R}_0^+$ and inverse elements don't generally exist. The non-existence of inverse elements to $+$ is the only reason why $(\mathbb{R}_0^+, +, \cdot)$ is not a linear space similarly to set of closed balls in inner product space.

□

These two examples have introduced structures that have very similar properties and the same problem of non-existence of inverse elements to $+$. For real numbers the sollution is well-known. Let's follow this well-known construction of negative numbers as equivalence classes upon the set of pairs and define some kind of negative sets.

Definition 2 *Let $\mathcal{P}(\mathbf{V})$ denotes the set of all subsets of the linear space $\mathbf{V}$. Denote by $\mathfrak{M}$ any subset of $\mathcal{P}(\mathbf{V})$ for which following conditions are satisfied:*

(M1) $\exists \mathbf{A} \in \mathfrak{M}, \mathbf{A} \neq \emptyset$

(M2) $\forall \mathbf{A}, \mathbf{B} \in \mathfrak{M} : \mathbf{A} + \mathbf{B} \in \mathfrak{M}$

(M3) $\forall \lambda \in \mathbb{R}_0^+, \mathbf{A} \in \mathfrak{M} : \lambda \mathbf{A} \in \mathfrak{M}$

(M4) $\forall \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathfrak{M} : \mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{C} \Rightarrow \mathbf{A} = \mathbf{B}$

(M5) $\forall \lambda, \alpha \in \mathbb{R}_0^+, \mathbf{A} \in \mathfrak{M} : (\lambda + \alpha)\mathbf{A} = \lambda \mathbf{A} + \alpha \mathbf{A}$

(M6) $\forall \mathbf{A} \in \mathfrak{M}, x \in \mathbf{V} : \mathbf{A} + x \in \mathfrak{M}$

(M7) $\forall \mathbf{A}, \mathbf{B} \in \mathfrak{M} : \exists \mathbf{C} \in \mathfrak{M}, o \in \mathbf{C} : \mathbf{A} \subset \mathbf{B} \Rightarrow \mathbf{A} + \mathbf{C} = \mathbf{B}$

Remark 1 An example of the set $\mathfrak{M}$ may be the set $\mathfrak{B}$ of all closed balls. In general, $(\mathfrak{M}, +)$ is a monoid with the cancellation property, For non-negative real numbers both $+, \cdot$ distribution laws hold, for all $\mathbf{A} \in \mathfrak{M}$: $1 \cdot \mathbf{A} = \mathbf{A}$ and as in previous examples inverse elements don't generally exist. The non-existence of inverse elements is the only reason why $(\mathfrak{M}, +, \cdot)$ is not a linear space.

Definition 3 *Let the set $\mathfrak{M}$ satisfies* **(M1)-(M7)**. *We define the binary relation $\sim$ upon the set $\mathfrak{M}^2$:*

$$[\mathbf{A}_1, \mathbf{A}_2] \sim [\mathbf{B}_1, \mathbf{B}_2] \Leftrightarrow \mathbf{A}_1 + \mathbf{B}_2 = \mathbf{A}_2 + \mathbf{B}_1$$

Lemma 1 *Relation $\sim$ is equivalence.*

Proof: The proof is straightforward. The cancellation property of $+$ is necessary to prove transitivity.

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Definition 4 *Let $\mathcal{V}(\mathfrak{M})$ denotes the set of all equivalence classes of the relation $\sim$. We define*

$$[\mathbf{A}_1, \mathbf{A}_2] + [\mathbf{B}_1, \mathbf{B}_2] = [\mathbf{A}_1 + \mathbf{B}_1, \mathbf{A}_2 + \mathbf{B}_2],$$

$$\lambda \cdot [\mathbf{A}_1, \mathbf{A}_2] = \begin{cases} [\lambda \cdot \mathbf{A}_1, \lambda \cdot \mathbf{A}_2] & \text{for } \lambda \geq 0, \\ [(-\lambda) \cdot \mathbf{A}_2, (-\lambda) \cdot \mathbf{A}_1] & \text{otherwise.} \end{cases}$$

Remark 2 It is necessary to prove that the previous definition is correct. The proof is straightforward and the cancellation property of $(\mathfrak{M}, +)$ and both distribution laws are essential.

Theorem 1 *$(\mathcal{V}(\mathfrak{M}), +, \cdot)$ is a linear space over $\mathbb{R}$.*

Proof: Straightforward.

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2 Structure of the space

Theorem 2 *Consider a mapping $\varphi \colon \mathfrak{M} \to \mathcal{V}(\mathfrak{M})$, $\varphi(\mathbf{A}) = [\mathbf{A}, \{o\}]$. Then φ is injective homomorphism of $(\mathfrak{M}, +, \cdot)$ into $(\mathcal{V}(\mathfrak{M}), +, \cdot)$.*

Definition 5 *The homomorphism φ defined above will be called* natural homomorphism *of $(\mathfrak{M}, +, \cdot)$ into $(\mathcal{V}(\mathfrak{M}), +, \cdot)$.*

The existence of the natural homomorphism φ allows us to identify elements of $\mathfrak{M}$ with corresponding elements of $\mathcal{V}(\mathfrak{M})$. These elements will be called *non-negative (or positive) sets*, the other elements will be called *negative sets*.

Theorem 3 *Consider a mapping $\vartheta \colon \mathbf{V} \to \mathcal{V}(\mathfrak{M})$, $\vartheta(x) = [\{x\}, \{o\}]$. Then ϑ is injective homomorphism of $(\mathbf{V}, +, \cdot)$ into $(\mathcal{V}(\mathfrak{M}), +, \cdot)$.*

Definition 6 *The homomorphism ϑ defined above will be called* natural homomorphism *of $(\mathbf{V}, +, \cdot)$ into $(\mathcal{V}(\mathfrak{M}), +, \cdot)$.*

The existence of the natural homomorphism ϑ allows us to identify elements of $\mathbf{V}$ with corresponding elements of $\mathcal{V}(\mathfrak{M})$. Images in ϑ of points in $\mathbf{V}$ will be called *points in $\mathcal{V}(\mathfrak{M})$*. For the structure of the space $\mathcal{V}(\mathfrak{M})$ see fig. 1.

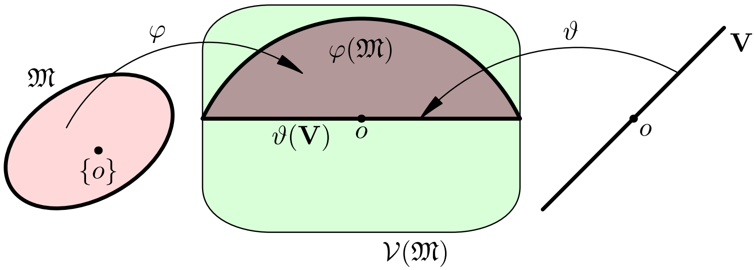

Figure 1: Structure of the space $\mathcal{V}(\mathfrak{M})$

3 Generalized subset relation

Now let's consider structure $(\mathfrak{M}, \subset)$ as the partially ordered set. We may expect homomorphism φ converts the relation $\subset$ to some newly defined partial order relation $\preceq$.

Definition 7 *Define the binary relation $\preceq$ upon the set $\mathcal{V}(\mathfrak{M})$:*

$$[\mathbf{A}_1, \mathbf{A}_2] \preceq [\mathbf{B}_1, \mathbf{B}_2] \Leftrightarrow \mathbf{A}_1 + \mathbf{B}_2 \subset \mathbf{A}_2 + \mathbf{B}_1$$

If for $A, B \in \mathcal{V}(\mathfrak{M})$ holds $A \preceq B$ we say that A is contained in *B.*

Remark 3 It is necessary to prove that the previous definition is correct. **(M4)** and **(M7)** are necessary.

Theorem 4 *Relation $\preceq$ is a partial order relation, i.e. it is reflexive, antisymmetrical and transitive.*

Following theorem allows us to consider relation $\preceq$ as a generalized subset relation.

Theorem 5 *Consider the natural homomorphism φ of $(\mathfrak{M}, +, \cdot)$ into $(\mathcal{V}(\mathfrak{M}), +, \cdot)$. Then $\mathbf{A} \subset \mathbf{B}$ if and only if $\varphi(\mathbf{A}) \preceq \varphi(\mathbf{B})$. It means that φ is an isomorphism of $(\mathfrak{M}, +, \cdot, \subset)$ onto $(\varphi(\mathfrak{M}), +, \cdot, \preceq)$.*

4 Example – Oriented ball space

In this section, we show a particular example of our geometry. If we start with set of all closed balls in inner product space we get space identical to classical oriented sphere space.

Definition 8 *Consider set* $\mathfrak{B} = \{B(x, r) : x \in \mathbf{X}, r \in \mathbb{R}_0^+\}$ *of all closed balls in* $\mathbf{X}$*, where* $\mathbf{X}$ *is the inner product space. Partial ordered linear space* $(\mathcal{V}(\mathfrak{B}), +, \cdot, \preceq)$ *will be called* oriented ball space.

Now, it's time to show relationship between oriented ball space and oriented sphere space. This relationship is obvious for non-negative balls. Denote by $\mathcal{S}(x, r)$ classical Laguerre's oriented sphere where $x \in \mathbf{V}$ denotes center of the sphere and $r \in \mathbb{R}$ denotes oriented radius, and by $\mathfrak{S}$ the set of all oriented spheres in $\mathbf{V}$.

Definition 9 *Let's define the mapping* $\mathcal{B}\colon \mathbf{V} \times \mathbb{R} \to \mathcal{V}(\mathfrak{B})$*:*

$$\mathcal{B}(x, r) = \begin{cases} [B(x, r), \{o\}] & \text{if } r \geq 0 \\ [\{x\}, B(o, -r)] & \text{otherwise} \end{cases}$$

The element $\mathcal{B}(x, r) \in \mathcal{V}(\mathfrak{B})$ *will be called* oriented ball with center x and oriented radius r.

Definition 10 *Define indefinite inner product upon the space* $\mathcal{V}(\mathfrak{B})$*:*

$$\Big\langle \mathcal{B}(x_1, r_1), \mathcal{B}(x_2, r_2) \Big\rangle_{PE} = \langle x_1, x_2 \rangle - r_1 r_2$$

Remark 4 The indefinite inner product from the previous definition is obviously indefinite inner product.

Theorem 6 *Consider mapping* $\alpha\colon \mathfrak{S} \to \mathcal{V}(\mathfrak{B})$, $\alpha(\mathcal{S}(x, r)) = \mathcal{B}(x, r)$*. Then* α *is the isomorphism of* $(\mathfrak{S}, +, \cdot)$ *onto* $(\mathcal{V}(\mathfrak{B}), +, \cdot)$ *and also isometry in corresponding indefinite inner product spaces.*

Proof: Obvious.

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Remark 5 The geometrical meaning of these two spaces is nearly the same and there is one-to-one "identical" mapping of each space to another that maps a sphere to ball with the same center and oriented radius. We may identify these two spaces.

5 Klein Geometry

Definition 11 *Let's consider a space* $\mathcal{V}(\mathfrak{M})$ *and denote by* $\mathfrak{L}(\mathcal{V}(\mathfrak{M}))$ *a group of all afinne transformations of the space* $\mathcal{V}(\mathfrak{M})$ *preserving the partial order relation* $\preceq$*. Then we define a geometry*

$$G = \Big(\mathcal{V}(\mathfrak{M}), \mathfrak{L}(\mathcal{V}(\mathfrak{M}))\Big)$$

It is obvious from above that a particular example of this new kind of geometry is similar to Laguerre geometry, given the set $\mathfrak{B}$ of all closed balls as the set $\mathfrak{M}$. Accurately, we may say that every Laguerre transformation (considered as similarity in indefinite inner product space) can be written as the transformation preserving $\preceq$ or composed mapping of transformaiton preserving $\preceq$ and a transformation that maps every oriented ball $\mathcal{B}(x,r)$ to $\mathcal{B}(x,-r)$.

6 Summary

This new approach to Laguerre geometry may allow us to use today's latest methods of this geometry in more general spaces to solve more general kind of problems. The result may be faster algorithm to solve the problem where another way is used today, easier proof of a theorem or maybe a way to solve some of the open problems.

Future work will be concentrated on a study of the system of conditions in definition 2, on a study of the spaces of the form $\mathcal{V}(\mathfrak{M})$, formulation of new theorems and on a study of well known problems of CAGD and application of this new geometry to solve them.

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