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AREA OF THE MINKOWSKI SUM OF TWO CONVEX SETS

Abstract

This paper deals with the area of the Minkowski sum of two convex polygons and the area of the Minkowski sum of two convex sets bounded by closed curves.

Keywords

Minkowski sum, area.

1 Introduction

The Minkowski sum of two point sets $\mathcal{A}$ and $\mathcal{B}$ can be defined as $\mathcal{A} \oplus \mathcal{B} = \bigcup\limits_{b \in \mathcal{B}} \mathcal{A}^{\boldsymbol{b}}$ or $\mathcal{A} \oplus \mathcal{B} = \{\boldsymbol{a} + \boldsymbol{b} |\ \boldsymbol{a} \in \mathcal{A} \wedge \boldsymbol{b} \in \mathcal{B}\}$.

We will find the rule for the computation of the area of the Minkowski sum of two convex sets.

2 Area of the Minkowski sum of two convex polygons

Theorem 1: (see fig. 1) Let $\mathcal{A}$ and $\mathcal{B}$ be convex polygons and $O = [0, 0]$ an inner point of the polygon $\mathcal{B}$ and $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$. Then the area $S(\mathcal{C})$ of the polygon $\mathcal{C}$ is

$$S(\mathcal{C}) = S(\mathcal{A}) + S(\mathcal{B}) + \sum_{i=1}^{n} |a_i||v_i|,$$

where $S(\mathcal{A}), S(\mathcal{B})$ are areas of the polygons $\mathcal{A}, \mathcal{B}$, a_i is the edge of the polygon $\mathcal{A}$, n_i is the outer normal vector of a_i and v_i is the distance of the extreme point in direction n_i on the polygon $\mathcal{B}$ from the straight line which is parallel with a_i and goes through the point $O = [0, 0]$.

Proof: (see fig. 2) Let $A_1, \dots, A_n$ be the vertices of the polygon $\mathcal{A}$ and $B_1, \dots, B_m$ the vertices of the polygon $\mathcal{B}$. We denote the

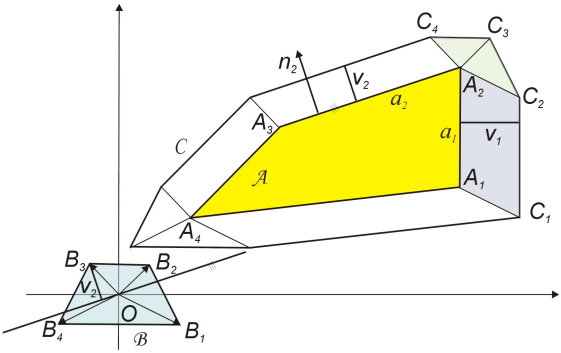

Figure 1: Area of the Minkowski sum of two convex polygons 1

position vectors of $B_1, \ldots, B_m$ as $\boldsymbol{b}_1, \ldots \boldsymbol{b}_m$ the edges of polygon $\mathcal{A}$ for $i = 1, \ldots, n$ $(A_{n+1} = A_1)$ as $a_i = A_iA_{i+1}$ and the edges of polygon $\mathcal{B}$ for $j = 1, \ldots, m$ $(B_{m+1} = B_1)$ as $b_j = B_jB_{j+1}$.

If $\mathcal{A}$ and $\mathcal{B}$ do not have parallel edges, then the number of edges of the Minkowski sum $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$ is $m + n$ $(m, n$ are the numbers of edges of the polygons $\mathcal{A}$ and $\mathcal{B})$. Each edge of the polygon $\mathcal{C}$ is parallel and equivalent to the edge of $\mathcal{A}$ or $\mathcal{B}$.

If any edge of $\mathcal{A}$ is parallel to the edge of $\mathcal{B}$, then the edge of $\mathcal{C}$ can be parallel to both these edges and its length is the sum of the lengths of these two edges. This situation can be converted to the previous case by the addition of an auxiliary vertex which divides this edge into two parts whose lengths are equivalent to the lengths of the edges of the polygons $\mathcal{A}$ and $\mathcal{B}$.

We denote the vertices of the polygon $\mathcal{C}$ as $C_1, \ldots, C_{m+n}$and its edges as $c_k^i = C_kC_{k+1}$ or $c_k^j = C_kC_{k+1}$. The superscript denotes to which edge of the polygon $\mathcal{A}$ or $\mathcal{B}$ the edge of $\mathcal{C}$ is parallel, i.e. $c_k^i = C_kC_{k+1}$ $(i = 1, \ldots, n)$ is parallel to the edge a_i of the polygon $\mathcal{A}$ and $c_k^j = C_kC_{k+1}$ $(j = 1, \ldots, m)$ is parallel to the edge b_j of the polygon $\mathcal{B}$.

We can divide the area of the polygon $\mathcal{C}$ into two parts. The area of $\mathcal{A}$ fills the first part of $\mathcal{C}$ $(\mathcal{A} \subset \mathcal{C}$ because $[0, 0] \in \mathcal{B})$. The triangles and parallelograms which we create in the following way fill the second

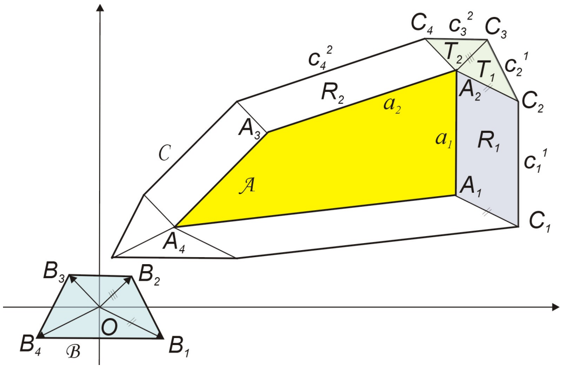

Figure 2: Area of the Minkowski sum of two convex polygons 2

part of $\mathcal{C}$.

1. There is an edge c_k^i to each edge a_i such that $a_i \parallel c_k^i$ and $|a_i| = |c_k^i|$. We can obtain c_k^i by moving the edge a_i by the position vector $\boldsymbol{b}_j$ of the extreme point B_j in direction n_i on the polygon $\mathcal{B}$ and $C_k = A_i + \boldsymbol{b}_j$.

   We can create the parallelogram R_i to each edge a_i such that the two edges of R_i are $a_i = A_iA_{i+1}$ and $c_k^i = C_kC_{k+1}$ ($a_i \parallel c_k^i$). The remaining two edges are equivalent and parallel to the position vectors of the extreme point B_j in direction n_i on the polygon $\mathcal{B}$. The area of the parallelogram R_i is $S(R_i) = |a_i||v_i|$ where v_i is the altitude of the parallelogram R_i to the edge a_i. It is the same as the distance of the extreme vertex in direction n_i on the polygon $\mathcal{B}$ from the straight line which is parallel to the edge a_i and goes through the point $O = [0, 0]$.

2. The edges $c_k^j = C_kC_{k+1}$ are parallel and equivalent to the edges of the polygon $\mathcal{B}$. Since $C_k = A_i + \boldsymbol{b}_j$ a $C_{k+1} = A_i + \boldsymbol{b}_{j+1}$ we can create the triangle $T_j = C_kC_{k+1}A_i$ to each edge c_k^j. The triangle T_j is equivalent to the triangle OB_jB_{j+1}. The sum of the areas of these triangles gives the area of the polygon $\mathcal{B}$. Thus $\sum_{j=1}^{m} S(T_j) = S(\mathcal{B})$.

The polygon $\mathcal{A}$, the parallelograms R_i and triangles T_j fill the whole polygon $\mathcal{C}$ because the only situations that can arise are as follows:

1. Both edges c_k^i and c_{k+1}^{i+1} are parallel to the edges of the polygon $\mathcal{A}$ and the parallelograms R_i and R_{i+1} have a common edge.

2. The edge c_k^i is parallel to the edge of edge of the polygon $\mathcal{A}$ and c_{k+1}^j is parallel to the edge of edge of the polygon $\mathcal{B}$ (or $c_k^j \parallel b_j$ and $c_{k+1}^i \parallel a_i$). Then the parallelogram and the triangle have a common edge.

3. Both edges c_k^j and c_{k+1}^{j+1} are parallel to the edges of the polygon $\mathcal{B}$ and then the triangles T_j and T_{j+1} have a common edge.

As the polygon $\mathcal{C}$ is convex, the triangles and parallelograms do not overlap.

We can express the area of the polygon $\mathcal{C}$ as the sum of the areas of the polygons $\mathcal{A}$ and $\mathcal{B}$ and the areas of the parallelograms R_i. Thus $S(\mathcal{C}) = S(\mathcal{A}) + S(\mathcal{B}) + \sum_{i=1}^{n} |a_i||v_i|$. ■

Remark: Since $\mathcal{A}^s \oplus \mathcal{B}^t = (\mathcal{A} \oplus \mathcal{B})^{s+t}$, the relation for the computation of the area of Minkowski sum holds for arbitrarily placed polygons.

2.1 Area of the Minkovski sum of two convex sets bounded by closed curves

Theorem 2: (see fig. 3) Let $\mathcal{A}, \mathcal{B}$ be the convex, bounded and closed sets in E_2 and $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$.

Let the curve $C_1(t) = (x_1(t), y_1(t)), t \in I$ be the boundary of the set $\mathcal{A}$ and the curve $C_2(s) = (x_2(s), y_2(s)), s \in J$ be the boundary of the set $\mathcal{B}$ and let the transformation $s(t) : I \to J$ of the parametr s be such that $(dx_1(t), dy_1(t)) = k(dx_2(s(t)), dy_2(s(t))), t \in I$ and $k > 0$. Then the area $S(\mathcal{C})$ of the set $\mathcal{C}$ is

$$S(\mathcal{C}) = S(\mathcal{A})+S(\mathcal{B})+\int_I \|(x_2(s(t)), y_2(s(t)), 0)\times(dx_1(t), dy_1(t), 0)\| \, dt \,. \tag{1}$$

Proof: We know that an extreme point in direction $\boldsymbol{d}$ on the set $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$ is the sum of the extreme points in direction $\boldsymbol{d}$ on the sets $\mathcal{A}$ and $\mathcal{B}$ (see [1]).

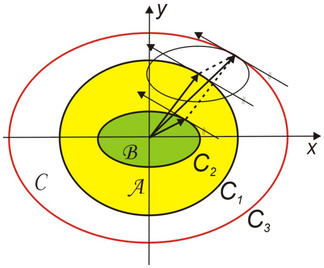

Figure 3: Area of the Minkowski sum of two convex sets bounded by closed curves

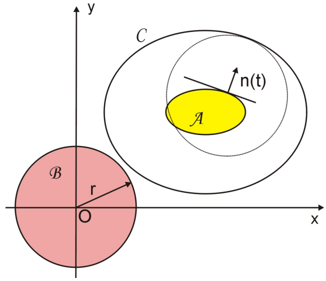

Figure 4: Area of the Minkowski sum where one set is bounded by a circle

For convex sets each point on the boundary is also an extreme point. This means that we obtain points on the boundary of the set $\mathcal{C}$ as the sum of the points in which $C_1(t)$ and $C_2(s)$ have the same unit outer normal vectors.

If we find the parametrization of the curve $C_2(s)$ such that $I \to J$ and
$(dx_1(t), dy_1(t)) = k(dx_2(s(t)), dy_2(s(t))), t \in I = \langle a, b\rangle$ and $k > 0$ then boundary C_3 of the set $\mathcal{C}$ is $C_3(t) = (x_1(t) + x_2(s(t)), y_1(t) + y_2(s(t)))$ (see [3]).

From the Green theorem it follows that the area of the face $S(\mathcal{C})$ whose boundary is the closed curve C_3 is

$$S(\mathcal{C}) = \oint_{C_3} x\ dy = \int_I (x_1(t) + x_2(s(t))\ d(y_1(t) + y_2(s(t))) =$$
$$= \int_I x_1(t)\ d(y_1(t)) + \int_I x_2(s(t)))\ d(y_2(s(t)))) + \int_I (x_2(s(t))\ d(y_1(t)) +$$
$$x_1(t)\ d(y_2(s(t)))) =$$
$$= S(\mathcal{A}) + S(\mathcal{B}) + \int_I (x_2(s(t))\ d(y_1(t)) + x_1(t)\ d(y_2(s(t)))).$$

With the help of per partes we obtain

$$\int_I x_1(t)\ d(y_2(s(t))) = [x_1(t)y_2(s(t))]_a^b - \int_I x_1(t)y_2(s(t))\ dt.$$

The curve $C_3(t)$ is bounded so $[x_1(t)y_2(s(t))]_a^b = 0$ and therefore

$$\int_I [x_2(s(t))\ d(y_1(t)) + x_1(t)\ d(y_2(s(t)))] =$$
$$= \int_I [x_2(s(t))\ d(y_1(t)) - dx_1(t)(y_2(s(t)))] =$$
$$= \int_I \|(x_2(s(t)), y_2(s(t)), 0) \times (dx_1(t), dy_1(t), 0)\|\ dt\ . \qquad \blacksquare$$

Example: (see fig. 4) Let the boundary of the set $\mathcal{B}$ be a circle of the radius r with the the centre in $O = [0,0]$. For the curve C_3 (boundary of the set $\mathcal{C}$) it holds that
$C_3(t) = C_1(t) + C_2(s(t)) = C_1(t) + rn(t)$, where $n(t)$ is the unit outer normal vector of the curve $C_1(t)$ thus

$$n(t) = \left(\frac{dy_1(t)}{\sqrt{dx_1^2(t)+dy_1^2(t)}}, \frac{-dx_1(t)}{\sqrt{dx_1^2(t)+dy_1^2(t)}}\right).$$

After substitution to expression (1) we obtain

$$S(\mathcal{C}) = S(\mathcal{A}) + S(\mathcal{B}) + \int_I \sqrt{dx_1^2(t) + dy_1^2(t)} = S(\mathcal{A}) + S(\mathcal{B}) + r\ d(C_1),$$

where $d(C_1)$ is the length of the curve C_1.

3 Conclusion

In this paper we have presented rules for computation of the area of the Minkowski sum of two convex sets. The estimate for the non convex sets is a problem for the future research.

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