Offset surfaces and their usage in the milling

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Abstract. The undercut in 3-axis and 5-axis milling causes serious problems because of unexpected damages of the machined surface. Possible solutions for indication of this problem in advance can be based on (general) offset surfaces. This paper presents an algorithm for indication of the undercut problem in the milling for surfaces with rational parametrization for which also a (general) offset surface has rational parametrization (so-called RC surfaces). The method uses algebraic geometry (Gröbner bases or Dixon (dialytic) resultant) and computation of a (general) offset surface self-intersection.

Keywords: Offset surface, general offset surface, convolution surface, RC surfaces, surface self-intersection.

1 Introduction

Milling is a process which is used in mechanical engineering to produce a surface of the desired shape. The path of the milling machine has to be planned on a so-called *(general) offset surface* which contains all position of a reference point on the milling machine axis during the milling.

There are two main types of milling: *3-axis milling* (a milling machine is able to do only translational movement, not rotational) and *5-axis milling* (a milling machine can do not only translational movement but also rotational movement around two different axes). One of the problems that occurs during milling (in both types, 3-axis and also 5-axis) is the so-called *undercut problem*. This means that during the movement around the desired surface the milling machine may cause irreversible damage to the already machined part of the desired surface.

One of the possible mathematical approaches to solve (or at least to indicate) this problem is based on the study of properties of (general) offset surfaces, especially properties of offset surfaces for the case of 5-axis milling and properties of general offset surfaces for the case of 3-axis milling.

Several papers have dealt with the undercut problem in milling, and even with the (general) offset surfaces. Wallner et al. discuss in [8] the self-intersections of offset curves and surfaces mainly from the differential geometry point of view and show how to determine maximum offset distance such that the offset does not neither locally nor globally self-intersect. Glaeser et al. in [4] then focus on finding conditions for collision free 3-axis milling of surfaces and also on the selection of cutting tools for a given surface using general offset surfaces and differential geometry. Wallner in [7] studies the connection between singularities and self-intersections of a general offset surface and the possible collision of milling machine with the desired surface during the motion, again mainly from the differential geometry point of view.

This paper is devoted to the connection of the undercut problem in milling to (general) offset surfaces and to identification (and/or computation) of self-intersections of these surfaces, especially for the special class of surfaces called *RC surfaces*. Thus, the main objective is to present solution to the indication of the undercut problem for all surfaces belonging to the class of RC surfaces.

2 Offset surfaces and general offset surfaces

The definition of an offset surface using a surface normal is well-known, that's why we focus only on general offset surfaces.

During the 3-axis milling the milling machine can perform only translational motion in directions of all three coordinate axes x, y, z in $\mathbb{R}^3$. Hence, without loss of generality we can simplify the situation such that from now X (representing the surface which we want to make) will be a smooth surface which can be represented in explicit form $z = f(x, y)$ and Σ (representing the cutter, i.e. milling machine) can be similarly represented in explicit form $z = e(x, y)$. Both these surfaces represent boundary of corresponding solids and they are oriented such that their normals has positive z-coordinate. Moreover, we will consider that Σ is strictly convex. Then the 3-axis milling means that the cutter Σ moves such that its axis is still parallel to z-axis of coordinate system and concurrently Σ always *touches* the surface X. This condition in fact corresponds to the so-called *general offset surface*.

Definition 1 *Let X and Σ be surfaces with the properties mentioned above. Let r be a reference point chosen on the axis of surface Σ and let $\tau(x;p)$, $x \in X$, $p \in \Sigma$ be a translation such that $\tau(x;\Sigma)$ touches X. Then the set*

$$S = \{\tau(x;r) | x \in X\}$$

is called the general offset surface *of X with respect to Σ.*

Hence, we can study properties of general offset surfaces in order to be able to indicate the undercut in 3-axis milling.

From the definition of a general offset surface it is not clear how to find it. The answer gives so-called *convolution surface/hypersurface* which has very close relation to the general offset surface, especially in case whether sets A and B and their boundaries are given as smooth surfaces/hypersurfaces.

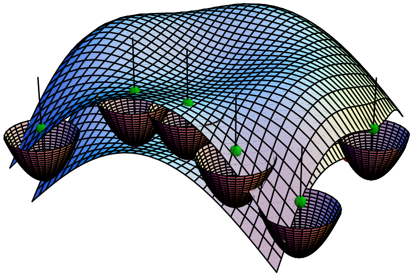

Figure 1: General offset surface (upper surface) of the Bézier surface (lower surface) with respect to the paraboloid

Definition 2 *Let A and B be smooth hypersurfaces in $\mathbb{R}^n$. Then the set*

$$C = A \star B = \{a + b | a \in A, b \in B \wedge t(a) \parallel t(b)\}$$

is called the convolution surface $A \star B$ *and $t(a)$, $t(b)$ are tangent hyperplanes of A and B at $a \in A$ and $b \in B$ which are called* corresponding points.

The reader can find more information on convolution surfaces e.g. in [5]. Here we only mention the connection of the convolution surface to general offset surfaces and 3-axis milling.

In fact, the general offset surface C can be computed as a convolution surface of the surface X and the cutter $-\Sigma$, i.e. $C = X \star (-\Sigma)$, where $-\Sigma$ represents so-called reflected cutter To illustrate defined notion, Fig. 1 displays the general offset surface obtained as the convolution surface of the Bézier surface BS with the paraboloid PS, i.e. it corresponds to the computation of the convolution surface $BS \star (-PS)$.

3 Solution of the undercut problem for RC surfaces

In the main part of this paper we focus on the solution of the undercut problem for a special class of so-called RC surfaces. The method is based on a computation (or at least indication) of a (general) offset surface self-intersection using algebraic geometry, especially variable elimination methods like Gröbner bases (see [2] for more details) or Dixon (Dixon dialytic) resultants (see [3] for more details). The presence of the self-intersection on a (general) offset surface *indicates* the undercut during the milling process.

3.1 RC surfaces

The aim of this subsection is to explain the first point of the algorithm presented in the next subsection, i.e. to give basic definitions and explanations of the notions *GRC hypersurface* and *SRC hypersurface* which together belong among *RC surfaces/hypersurfaces.* Let us start with exact definitions of these two classes of surfaces.

Definition 3 *Let A be a rational hypersurface in $\mathbb{R}^n$ parametrized by $\mathbf{a}(u_1, \ldots, u_{n-1})$. This parametrization is called a* GRC parametrization, *if and only if the convolution hypersurface $A \star B$ has an explicit rational parametrization for an arbitrary hypersurface B with rational parametrization. Further, A is called a* GRC hypersurface, *if and only if it possesses a GRC parametrization.*

Definition 4 *Let A be a rational hypersurface in $\mathbb{R}^n$ parametrized by $\mathbf{a}(u_1, \ldots, u_{n-1})$. This parametrization is called a* SRC parametrization, *if and only if there exists a hypersurface B with proper rational parametrization such that the convolution surface $A \star B$ has an explicit rational parametrization. Further, A is called a* SRC hypersurface, *if and only if it possesses a SRC parametrization.*

Roughly speaking, RC surfaces are surfaces with rational parametrization for which either

- the convolution surface with arbitrary rational surface is also rational — these RC surfaces are called *GRC surfaces* and typical example of GRC surfaces are *LN surfaces*, i.e. surfaces with linear normals (see [6] for more details). However, the class of GRC surfaces contain many other surfaces, not just LN surfaces;

or

- the convolution surface with at least one another rational surface (parametrized by proper parametrization) is rational — these RC surfaces are called *SRC surfaces* and typical example of SRC surfaces are *PN surfaces*, i.e. surfaces with pythagorean normal which yield rational convolution with sphere.

Let us remember here the close relation of the convolution surface and the (general) offset surface mentioned in Section 2, stating the identity between the general offset surface of surface X and cutter Σ and a convolution surface $C = X \star (-\Sigma)$. This means that *rational* general offset surface and offset surface are obtained *for all* GRC surfaces, independently on the surface representing the cutter, and *there exists* the surface representing the cutter for which the general offset surface is rational for all SRC surfaces (but not for all cutters).

As an example of GRC surfaces which certainly do not belong among LN surfaces we can name *non-quadric* (surfaces of degree 3 and 4) *quadratic polynomial parametric surfaces* — there exist 9 affine classes of such surfaces and all these classes belong among GRC surfaces (more details in [5]).

Interested reader can find more details on GRC and SRC surfaces in [5], for example how to decide if rational surface belongs among GRC surfaces, SRC surfaces or whether it does not belong in any of these classes.

3.2 Algorithm

This subsection is devoted to the main algorithm for identification of the undercut during the milling of RC surfaces.

Input: Parametrization X of the surface which should be machined, type of the milling (3-axis or 5-axis), parametrization E of boundary surface of the milling machine (only for 3-axis milling) or radius r_c of the ball cutter (for 5-axis milling).

Output: "The surface can be machined without any undercut" or "During milling the cutter damages the surface" or "We can not decide by this algorithm"

1. Decide if the input parametrization X is:
   1. GRC $\rightarrow$ Continue to Step 3,
   2. SRC $\rightarrow$ Continue to Step 2,
   3. Neither GRC, nor SRC $\rightarrow$ We can not decide, end.
2. **If the type of milling is 3-axis** $\rightarrow$ Check if the parametrization E of the milling machine provides the rational general offset surface of surface X. Answer is:
   1. Yes $\rightarrow$ Continue to Step 3,
   2. No $\rightarrow$ We can not decide, end.

   Otherwise Check if the surface X provides rational offset surface. Answer is:
   1. Yes $\rightarrow$ Continue to Step 3,
   2. No $\rightarrow$ We can not decide, end.
3. **If the type of milling is 3-axis** $\rightarrow$ Construct the general offset surface for surface X as a convolution surface $X \star (-E)$.
   Otherwise Construct the offset surface for surface X in distance r_c, either by definition or as a convolution surface $X \star S$ where S is a rational parametrization of sphere.
4. Compute surface self-intersection of (general) offset surface. Denote the set of points containing the self-intersection points by Y.
5. **If** $Y \equiv \emptyset \rightarrow$ The surface can be machined without any undercut.
   Otherwise During milling the cutter damages the surface.

The details of the subalgorithm mentioned in point 4 of the main algorithm for the computation of the self-intersection of surfaces with rational parametrization can be found in [1].

4 Conclusion

This paper has presented one possible solution to the identification of the undercut problem in 3-axis and 5-axis milling for surfaces with rational parametrization for which also the (general) offset surface is rational. As an example we have mentioned that among GRC surfaces belong all non-quadric quadratic polynomial parametric surfaces. It is a challenge for further research to prove that also at least some bicubic polynomial/rational parametric surfaces belong among GRC or SRC surfaces because they are usually used in CAD and CAM systems. At this point we can only say that unfortunately neither all bicubic polynomial parametric surfaces nor bicubic rational parametric surfaces belong among GRC surfaces.

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References

[1] Bastl, B.: Self-intersection of Curves and Surfaces. In Proceedings of GCG 2003, Hojsova Straz, Czech Republic, 2003.

[2] Becker, T., Weispfenning, V.: Gröbner bases - A Computational Approach to Commutative Algebra. Graduate Texts in Mathematics. Springer Verlag, New York, 1993.

[3] Chtcherba, A.D., Kapur, D.: Constructing Sylvester-type resultant matrices using the Dixon formulation. Journal of Symbolic Computation, Vol. 38, No. 1, pp. 777-814, Elsevier, 2004.

[4] Glaeser, G., Wallner, J., Pottmann, H.: Collision Free 3-Axis Milling and Selection of Cutting-Tools. Computer-Aided Design, Vol. 31, No. 3, pp. 225-232, Elsevier, 1999.

[5] Lávička, M., Bastl, B.: Rational Parametrized Curves and Surfaces with Rational Convolutions. In: Proceedings of AGGM 2006, Barcelona, Spain, 2006.

[6] Sampoli, M.L., Peternell, M., Jüttler, B.: Rational surfaces with linear normals and their convolutions with rational surfaces. Computer Aided Geometric Design, Vol. 23, No. 2, pp. 179-192, Elsevier, 2006.

[7] Wallner, J.: Self-Intersections and Smoothness of General Offset Surfaces. Journal of Geometry, Vol. 70, No. 1-2, pp. 176-190, Birkhuser Basel, 2001.

[8] Wallner, J., Sakkalis, T., Maekawa, T., Pottmann, H., Yu, G.: Self-Intersections of Offset Curves and Surfaces. International Journal of Shape Modeling, Vol. 7, No.1, pp. 1-21, World Scientific Publishing, 2001.