Computing exact offset surfaces of quadratic triangular Bézier patches

Bohumír Bastl[1]

(joint work with B. Jüttler[2], J. Kosinka[2], M. Lávička[1])

[1] Department of Mathematics, Faculty of Applied Sciences
University of West Bohemia, Plzen
Czech Republic

[2] Institute of Applied Geometry
Johannes Kepler University, Linz
Austria

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Outline

Motivation

- The importance of offset and general offset curves and surfaces follows from engineering practice, mainly from milling where two basic approaches are distinguished:
  - 5-axis milling, where crucial role play offset surfaces,
  - 3-axis milling, where general offset surfaces are especially important.
- CAD and CAM systems typically use as main objects NURBS curves and surfaces, i.e., curves and surfaces with polynomial/rational parametrization.
- Since the existence of rational parametrization of an object does not guarantee the rationality of offsets and general offsets, classes of curves and surface with rational offsets are especially important (LN-, GRC-, PH-, PN-), because:
  - it simplifies the representation of offsets,
  - it simplifies further computation with offsets, e.g. computation of self-intersection of offsets

Preliminaries

Quadratic triangular Bézier patch

A quadratic triangular Bézier patch is given by

$$\mathbf{a}(u, v, w) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=2}} \mathbf{p}_{ijk} \frac{2}{i!j!k!} u^i v^j w^k, \quad (1)$$

where $u, v, w \geq 0$ and $u + v + w = 1$.

Remark

- Since we will use Gaussian images of these patches, we exclude all developable surfaces from further considerations.

Gaussian map, parabolic curves and singular points

- Points where $\mathbf{a}_u \times \mathbf{a}_v$ vanishes are called singular points of $\mathbf{a}$, outside the singular locus the (unit) normal vector is defined everywhere.
- The Gaussian image of a surface (patch) has singular points along the so-called parabolic points of the surface which can be found by solving

$$(\mathbf{N} \cdot \mathbf{a}_{uu})(\mathbf{N} \cdot \mathbf{a}_{vv}) - (\mathbf{N} \cdot \mathbf{a}_{uv})^2 = 0. \qquad (2)$$

- The parabolic locus is a collection of planar curves – parabolas and (parts of) straight lines.
- Singular points of quadratic triangular patches correspond to common points of parabolic curves on a surface.

Offset surfaces

Definition

Let $A \subset \mathbb{R}^3$ be a smooth surface parametrized by $\mathbf{a}(u, v)$, $(u, v) \in \Omega \subset \mathbb{R}^2$ and let $\mathbf{n}(u, v), (u, v) \in \Omega$, be a normal vector. Then

$$\mathbf{a}_{OFF}(u, v) = \mathbf{a}(u, v) + d \cdot \frac{\mathbf{n}(u, v)}{||\mathbf{n}(u, v)||}, \quad (u, v) \in \Omega \qquad (3)$$

gives the offset surface parametrization of surface the A at the distance d.

Basic facts

- Offset surface defined by (3) is rational iff $||\mathbf{n}(u, v)||^2 = \sigma^2$, i.e., $\mathbf{a}(u, v)$ is a PN-parametrization.
- However, there no PN-parametrizations among parametrizations of quadratic triangular Bézier patches.
- Another approach is to use the so-called convolution surface.

Convolution surface

Definition

Let A and B be smooth surfaces in the affine space $\mathbb{R}^3$. The convolution surface $C = A \star B$ is defined as

$$C = \{\mathbf{a} + \mathbf{b} \,|\, \mathbf{a} \in A, \mathbf{b} \in B \text{ and } \alpha(\mathbf{a}) \parallel \beta(\mathbf{b})\}, \tag{4}$$

where $\alpha(\mathbf{a})$ and $\beta(\mathbf{b})$ are the tangent planes of A and B at points $\mathbf{a} \in A$ and $\mathbf{b} \in B$. The points $\mathbf{a}$, $\mathbf{b}$ are called corresponding points.

Remarks

- The parametric representation of the convolution surface $C = A \star B$ is given by

$$\mathbf{c}(s,t) = \mathbf{a}\left(\varphi_1(s,t), \varphi_2(s,t)\right) + \mathbf{b}(s,t),\ (s,t) \in \widetilde{D}_B. \tag{5}$$

General reparametrization formula

Theorem ([1])

Consider a non-developable quadratically parameterized surface A given by $\mathbf{a}(u,v) = \mathbf{a}_{20}u^2 + \mathbf{a}_{11}uv + \mathbf{a}_{02}v^2 + \mathbf{a}_{10}u + \mathbf{a}_{01}v + \mathbf{a}_{00}$. We denote

$$\mathbf{D} = (d_{ij}), \quad \mathbf{D}^u = (d^u_{ij}), \quad \mathbf{D}^v = (d^v_{ij}),\ i,j = 1,2,3, \quad \text{where} \tag{6}$$

$$d_{ij} = \begin{vmatrix} 2a_{20i} & a_{11i} \\ a_{11j} & 2a_{02j} \end{vmatrix},\ d^u_{ij} = \begin{vmatrix} a_{11i} & a_{10i} \\ 2a_{02j} & a_{01j} \end{vmatrix},\ d^v_{ij} = \begin{vmatrix} a_{10i} & 2a_{20i} \\ a_{01j} & a_{11j} \end{vmatrix}. \tag{7}$$

Let $\mathbf{n}_B = (\beta_1, \beta_2, \beta_3)^\top(s,t)$ be a normal vector of the surface B at the point $\mathbf{b}(s,t)$ satisfying the condition $\mathbf{n}_B^\top \mathbf{D} \mathbf{n}_B \neq 0$. Then the surface normals of A and B at the points $\mathbf{a}(u(s,t), v(s,t))$ and $\mathbf{b}(s,t)$, where

$$u = \frac{\mathbf{n}_B^\top \mathbf{D}^u \mathbf{n}_B}{\mathbf{n}_B^\top \mathbf{D} \mathbf{n}_B}, \quad v = \frac{\mathbf{n}_B^\top \mathbf{D}^v \mathbf{n}_B}{\mathbf{n}_B^\top \mathbf{D} \mathbf{n}_B}, \tag{8}$$

are parallel.

General reparametrization formula - remarks

- The reparametrization formula can be used for all quadratically parameterized surfaces except developable ones. In particular,
  - $\mathbf{D}$ is a zero matrix for parabolic cylinders,
  - $\mathbf{D}^u$ and $\mathbf{D}^v$ are zero matrices for a cone.
- If the denominator in reparametrization formula is not identically equal to zero, there can exist nonzero vectors such that this denominator vanishes – in this case, a regular point of B with $\mathbf{n}_B$ fulfilling $\mathbf{n}_B^\top \mathbf{D} \mathbf{n}_B = 0$
  - has no corresponding point on the quadratically parameterized surface A,
  - corresponds to parabolic points of quadratically parameterized surface A.
- Using reparametrization formula, the convolution surface has in this case the rational parametrization

$$\mathbf{c}(s,t) = \mathbf{a}\left(\frac{\mathbf{n}_B^\top \mathbf{D}^u \mathbf{n}_B}{\mathbf{n}_B^\top \mathbf{D} \mathbf{n}_B}, \frac{\mathbf{n}_B^\top \mathbf{D}^v \mathbf{n}_B}{\mathbf{n}_B^\top \mathbf{D} \mathbf{n}_B}\right) + \mathbf{b}(s,t),$$

Overview of the offset algorithm

Input: Non-developable quadratic Bézier patch P, distance d.

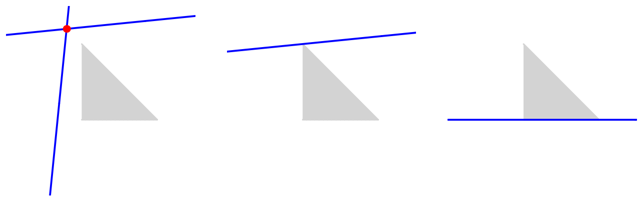
Output: Exact rational offset surface CS of patch P at distance d represented by pairs $\{\mathbf{cs}, D_{\mathbf{cs}}\}$.

1. Subdivide P along parabolic curves to at most 7 subpatches.
2. For each subpatch obtained in Step 1 do
   1. find a covering patch R of the subpatch Gaussian image on $\mathbb{S}^2$,
   2. with the help of reparametrization formula, compute the rational parametrization $\mathbf{cs}$ of the offset surface at distance d,
   3. find exact parametric domain $D_{\mathbf{cs}}$ of the offset surface for patch trimming
3. Return the collection of pairs $\{\mathbf{cs}, D_{\mathbf{cs}}\}$.

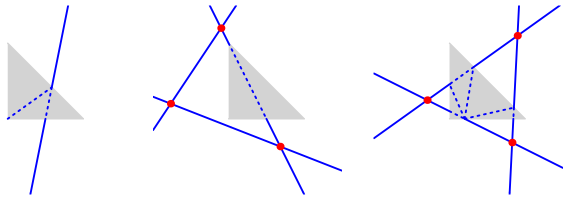
Algorithm 1: Subdividing the domain

Let $\mathcal{P}$ be the set of all preimages of parabolic points on P. Then:

1. $\mathcal{P} \cap \mathrm{int}(\Delta) = \emptyset$ . . . no subdivision is required.



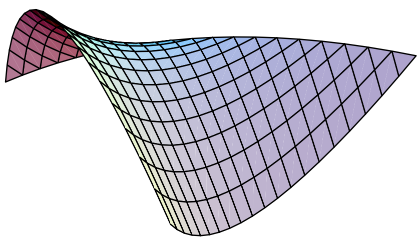
2. $\mathcal{P} \cap \mathrm{int}(\Delta) \neq \emptyset$ . . . the domain Δ is subdivided along the lines of $\mathcal{P}$ which intersect $\mathrm{int}(\Delta)$.

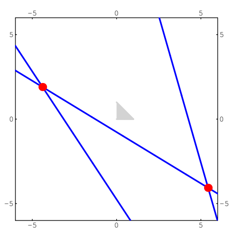


Algorithm 1: Examples I

Example 1 Patch given by

$$\begin{aligned}\mathbf{p}(u,v) \;=\; & \Big(\tfrac{1}{15}(3u^2 - u(7v+3) - 5(2v^2+v-3)),\\ & \tfrac{1}{5}(3u^2 + 2u(v-4) + 5),\\ & \tfrac{1}{15}(-24u^2 + u(27-7v) + 11v^2 - 8v - 3)\Big)^\top,\\ & (u,v) \in \Delta\end{aligned}$$

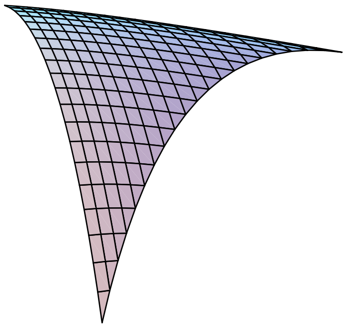


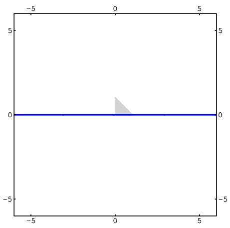


Algorithm 1: Examples II

Example 2 Patch given by

$$\mathbf{p}(u,v) = \left(u, u^2+v, v^2\right)^\top, \ (u,v) \in \Delta.$$





Algorithm 2: Covering the Gaussian image I

For each subpatch P given by $\mathbf{p}$ and obtained in Algorithm 1 we compute the covering patch R of the Gaussian image $\Gamma(\mathbf{p})$ on $\mathbb{S}^2$ using the following steps:

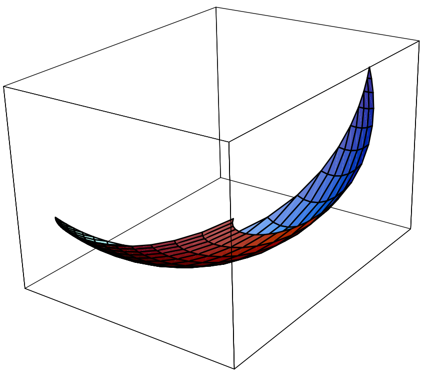
- Let $\mathcal{P}$ be the set of all preimages of parabolic points on P.
- If $\mathcal{P} \cap \Delta$ is not a line segment, then
    - the Gaussian image $\Gamma(\mathbf{p})$ is a spherical triangle
    - choose a suitable pole $\mathbf{z}$ for the stereographic projection $\sigma_{\mathbf{z}}$ onto the plane $\pi_{\mathbf{z}}$
    - project $\Gamma(\mathbf{p})$ to $\pi_{\mathbf{z}} \longrightarrow \Omega = \sigma_{\mathbf{z}}(\Gamma(\mathbf{p}))$
    - construct a circumscribed triangle of $\Omega \longrightarrow \widehat{\Omega}$
    - describe $\widehat{\Omega}$ as the linear Bézier triangle over $\Delta \longrightarrow \omega(s,t)$
    - project $\omega(s,t)$ back on $\mathbb{S}^2$ using inverse stereographic projection $\sigma_{\mathbf{z}}^{-1} \longrightarrow \mathbf{r}(s,t)$

Algorithm 2: Covering the Gaussian image II

- Else
  - the Gaussian image $\Gamma(\mathbf{p})$ is a spherical biangle
  - choose a suitable pole $\mathbf{z}$ for the stereographic projection $\sigma_{\mathbf{z}}$ onto the plane $\pi_{\mathbf{z}}$
  - project $\Gamma(\mathbf{p})$ to $\pi_{\mathbf{z}} \longrightarrow \Omega = \sigma_{\mathbf{z}}(\Gamma(\mathbf{p}))$
  - construct a circumscribed angle of $\Omega \longrightarrow \widehat{\Omega}$
  - describe $\widehat{\Omega}$ as the rational linear Bézier triangle over Δ with two vertices at infinity $\longrightarrow \omega(s,t)$
  - project $\omega(s,t)$ back on $\mathbb{S}^2$ using inverse stereographic projection $\sigma_{\mathbf{z}}^{-1} \longrightarrow \mathbf{r}(s,t)$
- The output of the algorithm is the rational patch R given by the parametrization $\mathbf{r}(s,t)$ over Δ.

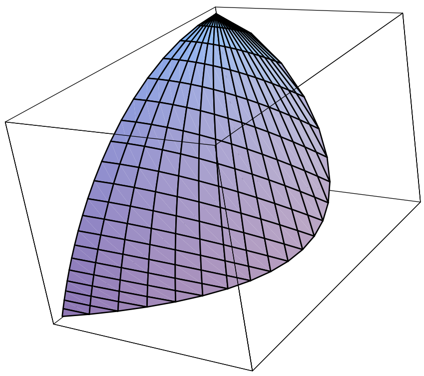
Algorithm 2: Examples - continued

Example 1

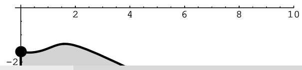


$\mathbf{z} = (0,0,1)^\top$, $\pi_{\mathbf{z}} : z = 0$

Example 2



$\mathbf{z} = (0,0,1)^\top$, $\pi_{\mathbf{z}} : z = 0$



Algorithm 3: Parametrization of offset and trimming

For each **p** obtained in Algorithm 1 and **r** obtained in Algorithm 2 we find a parametrization of the offset surface and exact parametric domain.

- Substitute numerator of **r** for normal vector $\mathbf{n}_B$ in the general reparametrization formula

$$u_{\mathbf{r}} = \frac{\mathbf{r}_n^\top \mathbf{D}^u \mathbf{r}_n}{\mathbf{r}_n^\top \mathbf{D} \mathbf{r}_n}, \qquad v_{\mathbf{r}} = \frac{\mathbf{r}_n^\top \mathbf{D}^v \mathbf{r}_n}{\mathbf{r}_n^\top \mathbf{D} \mathbf{r}_n}.$$

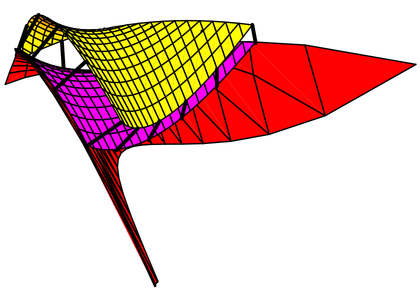
- Compute the offset surface parametrization

$$\mathbf{cs}(s,t) = \mathbf{p}(u_{\mathbf{r}}(s,t), v_{\mathbf{r}}(s,t)) + d\mathbf{r}(s,t),$$

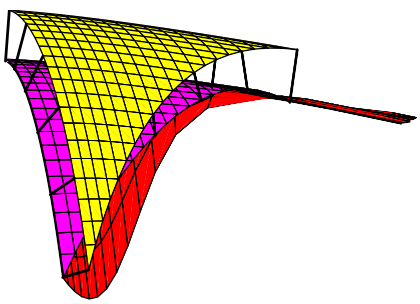
where $(s,t) \in \Delta$.

Algorithm 3: Examples - continued

Example 1



Example 2



Algorithm 3: Parametrization of offset and trimming II

Find the exact domain $\widetilde{D}_{\mathbf{r}} \subset \Delta$ such that for all $\mathbf{n}_{\mathbf{r}}(s,t)$, $(s,t) \in \widetilde{D}_{\mathbf{r}}$, there exists a parallel $\mathbf{n}_{\mathbf{p}}(u,v), (u,v) \in \Delta$, on $\mathbf{p}$.

- Construct normal cones of boundary curves of patch $\mathbf{p}$

  $$\mathbf{C}_1(p,q) = q\,\mathbf{n}(0,p),\ \mathbf{C}_2(p,q) = q\,\mathbf{n}(p,0),\ \mathbf{C}_3(p,q) = q\,\mathbf{n}(p,1-p).$$

- Find implicit representations $F_i(x,y,z) = 0$ of these cones.
- Find intersection curves $\mathbf{C}_i \cap \mathbb{S}^2$ projected by $\sigma_{\mathbf{z}}$ onto $\pi_{\mathbf{z}}$ by elimination of variables from

$$\begin{aligned}
x^2+y^2+z^2-1 &= 0,\\
F_i(x,y,z) &= 0,\\
\xi(1-\mathbf{x}\cdot\mathbf{z})-(x-z_1) &= 0,\\
\eta(1-\mathbf{x}\cdot\mathbf{z})-(y-z_2) &= 0,\\
\mu(1-\mathbf{x}\cdot\mathbf{z})-(z-z_3) &= 0,\\
1-w(1-\mathbf{x}\cdot\mathbf{z}) &= 0.
\end{aligned}$$

Algorithm 3: Parametrization of offset and trimming III

- Choose a proper sign of each $f_i \longrightarrow \bar{f}_i$
- Domain $\Omega = \sigma_{\mathbf{z}}(\Gamma(\mathbf{p}))$ is described as

$$\bigcap_i \bar{f}_i(\xi,\eta) \geq 0.$$

- Transform domain Ω into Δ to obtain final parametric domain $\widetilde{D}_{\mathbf{r}}$ of the offset surface

  $$\tau(\Omega) = \widetilde{D}_{\mathbf{r}}$$

  by transforming polynomials $\bar{f}_i$

  $$g_i(s,t) = \tau(\bar{f}_i(s,t)).$$

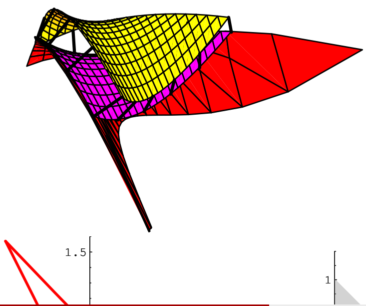
- The parametric domain $\widetilde{D}_{\mathbf{r}}$ is described as

$$\widetilde{D}_{\mathbf{r}} = \{(s,t) \in \Delta\ :\ g_i(s,t) \geq 0,\ i = 1,\ldots,k\}.$$

Algorithm 3: Examples - closure

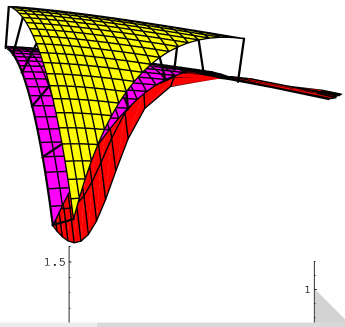
Example 1

- Reparametrization $(u_{\mathbf{r}}(s,t), v_{\mathbf{r}}(s,t)$ is of degree 4.
- Convolution surface $\mathbf{cs}(s,t)$ is of degree 10.



Example 2

- Reparametrization $(u_{\mathbf{r}}(s,t), v_{\mathbf{r}}(s,t)$ is of degree 4.
- Convolution surface $\mathbf{cs}(s,t)$ is of degree 8.



Summary

- The algorithm for computing exact rational offset surface of a quadratic triangular Bézier surface was presented.
- Algorithm is based on the fact that for all non-developable quadratic triangular Bézier surfaces the convolution with an arbitrary rational surface is also rational, i.e., offset surfaces are rational.
- Algorithm uses the Gaussian image of a patch and its stereographic projection for finding circumscribing rational patch on the sphere which is used in convolution.
- Offset surface is given by a rational parametrization and is of degree at most 10.
- Exact parametric domain given by at most three implicit functions in parametric plane is found.

For Further Reading I

- Bastl, B., Jüttler, B., Kosinka, J., Lávička, M.
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- Sampoli, M.L., Peternell, M., Jüttler, B.
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